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On Finding Integer Solutions To Non-homogeneous Ternary Cubic Equation

$$x^2 + b y^2 = (m^2 + b n^2) z^3$$

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Abstract:

The purpose of this paper is to obtain different sets of non-zero distinct integral solutions of ternary non-homogeneous cubic Diophantine equation $x^2 + b y^2 = (m^2 + b n^2) z^3$. A few properties of interest are presented.

Key words: Cubic with three unknowns ,Non-homogeneous cubic ,Integer solutions

Notations:

- T_n -Triangular number of rank n
- HP_n -Heptagonal number of rank n
- O_n -Octagonal number of rank n
- D_n -Decagonal number of rank n
- DD_n -Dodecagonal number of rank n
- Th_n -Tetrahedral number of rank n
- PP_n -Pentagonal Pyramidal number of rank n

Introduction:

The theory of Diophantine equations offers a rich variety of fascinating problems. In particular, cubic diophantine equations, homogeneous and non-homogeneous have aroused the interest of numerous mathematicians since antiquity [1-4]. In this context, one may refer [5-25] for various problems on the cubic diophantine equations with three variables, where, in each of the problems, different sets of non-zero integer solutions are obtained. However, often we come across homogeneous and non-homogeneous cubic equations and as such one may require its integral solution in its most general form. It is towards this end, this paper concerns with the problem of determining a general form of non-trivial integral solutions of the non-homogeneous cubic equation with three unknowns given by $x^2+by^2=(m^2+bn^2)z^3$.

Method of Analysis :

The non-homogeneous ternary cubic diophantine equation under consideration is

$$x^2+by^2=(m^2+bn^2)z^3 \quad (1)$$

where $b\neq 0$ and m,n are arbitrary integers not equal to zero simultaneously.

Taking

$$z=p^2+bq^2,\ p,q\neq 0 \quad (2)$$

in (1), it is written as

$$(x+i\sqrt{b}y)(x-i\sqrt{b}y)=(m+i\sqrt{b}n)(m-i\sqrt{b}n)(p+iq\sqrt{b})^3(p-iq\sqrt{b})^3 \quad (3)$$

Assume

$$x+i\sqrt{b}y=(m+i\sqrt{b}n)(p+i\sqrt{b}q)^3 \quad (4)$$

Since complex conjugates occur in pairs, we have

$$x-i\sqrt{b}y=(m-i\sqrt{b}n)(p-i\sqrt{b}q)^3 \quad (5)$$

From (4) and (5), the values of x and y are obtained as

$$x=m(p^3-3bpq^2)-bn(3p^2q-bq^3) \quad (6)$$

$$y=m(3p^2q-bq^3)+n(p^3-3bpq^2) \quad (7)$$

Thus, equations (2), (6) and (7) represent the integral solutions of (1).

To obtain various characterizations of solutions of (1), we have to consider it when the parameters m,n and b take particular values.

For illustration, the choice

$$m=1,\quad n=0,\quad b=-1 \quad (8)$$

in (1) leads to the diophantine equation

$$x^2-y^2=z^3 \quad (9)$$

We present various patterns of solutions of (9) below:

Pattern I

Substituting (8) in (2), (6) and (7), the values of x, y and z satisfying (9) are obtained as

$$x = p(p^2 + 3q^2)$$
$$y = q(3p^2 + q^2)$$
$$z = p^2 - q^2$$

A few interesting properties observed from the above solutions are given below:

1) Each of the expressions $x \pm y$, $4(x - az)(y + bz)$, $4(x + 3az)(y - 3bz)$ is a Cubical integer.

2) If the parameters (p, q) are taken as the squares of legs of a Pythagorean triangle the sum $x + y$ is a sextic integer.

3) Taking the parameter p to represent square of the hypotenuse and q to denote the square of a leg of a Pythagorean triangle, the difference $x - y$ is a sextic integer.

4) The triplet $(4pqz,\ 8p^2q^2,\ qx + py)$ forms a Pythagorean triplet

5) $py - qx = 2pqz$

6) Introducing the notations $x(p,q), y(p,q)$ and $z(p,q)$ for the solutions x, y and z respectively, the following relations are noticed.

a) $3[x(1,1+2q) - 2x(1,1+q) + 4x(1,q)]$ is a Nasty number.

b) $x(1,q) + 3z(1,q) = 4$

c) $2z(p+2q,q) - 4z(p+q,q) + 2z(p,q) = x(1,q) - z(1,q)$

d) $z(p+2q,q) - 2z(p+q,q) + z(p,q) = x(1,q) + z(1,q) - 2$

e) $z(p,q)z(p+q,p-q) = 2(2pq)z(p,q) = 2(py(p,q) - qx(p,q))$

f) $64x^2(p,1) = [y(p,1) + z(p,1)][y(p,1) + z(p,1) + 12]^2$

g) $z(p+q,q) - z(p-q,q) = 4pq = z(p+q,p-q)$

h) If $p, q (p > q)$ are taken as the generators of a Pythagorean triangle, then $z(p+q,q) + z(p-q,q) - z(p,q) = p^2 + q^2$ represents the hypotenuse of the Pythagorean triangle.

Pattern II

Introduction of the transformations

$$x = my,\ z = (m^2 - 1)\alpha^2,\ m > 1,\ \alpha \neq 0 \qquad (10)$$

in (9), simplifies it to

$$y = (m^2 - 1)\alpha^3 \qquad (11)$$

and thus

$$x = m(m^2 - 1)\alpha^3 \qquad (12)$$

The values of x, y and z presented in (10), (11) and (12) represent the solutions of (9).

Properties

1) $x = 6\alpha^3 Th_{m-1},$

2) $\frac{xz}{y} = 6Th_m$

3) The value of x may be considered to represent cubic multiple of the area of the Pythagorean triangle $(2m, m^2 - 1, m^2 + 1)$.

4) Each of the expressions $\frac{x^2}{x^2 - z^3}, \frac{mxy}{z^2}$ is a perfect square.

5) Each of the expressions $\frac{xy^2}{z^3}, \frac{xy}{z^2}$ is an integer.

6) $\frac{m^2 xy^2}{z^3}$ is a cubic integer.

7) $x + my - 2m\alpha z = 0$

8) $x^3 + m^3 y^3 - 8m^3\alpha^3 z^3 + 6xyzm^2\alpha = 0$

9) The triplet $(x,\ m\alpha z,\ my)$ forms an Arithmetic Progression.

10) $\alpha(mx - y) = z^2$

11) $m^3\alpha^3 x^3 - \alpha^3 y^3 - z^6 = 3m\alpha^2 xyz^2$

Pattern III

Assume the values of x and y to be

$$\begin{aligned} x &= 2^{3\alpha-2} v^{3\alpha-1} + v \\ y &= 2^{3\alpha-2} v^{3\alpha-1} - v \end{aligned} \qquad (13)$$

where α, v are non-zero constants.

From (9), we get

$$z = (2v)^{\alpha} \qquad (14)$$

Thus (13) and (14) represent the solutions of (9).

Properties

1) The triplet $(y,\ 2^{2(\alpha-1)} v^{2\alpha-1} z,\ x)$ forms an Arithmetic Progression with common difference v
2) Each of the expressions $2(x^2 - xy) - z^3, z^3 - 2(xy - y^2)$ is a perfect square.
3) Each of the expressions $12(x^2 - xy) - 6z^3,\ 6z^3 + 12(y^2 - xy)$ represents Nasty Number.
4) $x + y + z = (x - y)^{3\alpha-1} + (x - y)^{\alpha}$
5) $z = (x - y)^{\alpha}$

Pattern IV

Write (9) as

$$x - y = 1 \qquad (15)$$

$$x + y = z^3 \qquad (16)$$

Solving (15) and (16), we have

$$x = \frac{z^3 + 1}{2}, \qquad y = \frac{z^3 - 1}{2}$$

For x, y to be integers, z should be odd.

Thus taking $z = 2k - 1,\ (k \neq 1)$, the values of x and y are obtained as

$$x = 4k^3 - 6k^2 + 3k$$
$$y = 4k^3 - 6k^2 + 3k - 1$$

The above values of x, y and z represent the solutions of (9).

Properties

1) The solutions are primitive.

2) $2x - z - 1 = 6Th_{2k-1}$

3) $2x + z^2 - 1 = 2PP_{2k-1}$

4) $x - 18Th_k - 12T_k$ is a cubic number

5) $6HP_k - 18T_{k-1} - x \equiv 0 \pmod{5k}$

6) $6HP_k - 18T_{k-1} - y \equiv 0 \pmod{5k+1}$

7) $x - 8PP_k + 6T_{k-1} \equiv 0 \pmod{7}$

8) $y - 8PP_k + 6T_{k-1} \equiv -1 \pmod{7}$

9) $x = k(D_k) - 6T_{k-1}$

10) $y - k(D_k) + 1 \equiv 0 \pmod{6}$

11) $8PP_k - 2T_{k-1} - O_k - x$ is a Nasty number.

12) $x + 2T_k + DD_k \equiv 0 \pmod{4}$

13) $2(x + 2T_k + DD_k)$ is a cubical integer.

14) $6k(x + 2T_k + DD_k)$ is a Nasty number.

Conclusion:

In this paper, we have made an attempt to find non-zero distinct integer solutions to the non-homogeneous cubic equation with three unknowns given by $x^2 + by^2 = (m^2 + bn^2)z^3$. To conclude, one may search for other choices of general form of integer solutions to the cubic equation with three unknowns in title.

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